Relaxation and Lyapunov time scales in a one-dimensional gravitating sheet system

Toshio Tsuchiya*
Department of Astronomy, Kyoto University, Kyoto, 606-8502, Japan

Naoteru Gouda
National Astronomical Observatory, Mitaka 181-8588, Japan

(Received 2 August 1999; revised manuscript received 13 September 1999)

The relation between relaxation, the time scale of Lyapunov instabilities, and the Kolmogorov-Sinai time in a one-dimensional gravitating sheet system is studied. Both the maximum Lyapunov exponent and the Kolmogorov-Sinai entropy decrease as proportional to $N^{-1/5}$. The time scales determined by these quantities evidently differ from any type of relaxation time found in the previous investigations. The relaxation time to quasiequilibria (microscopic relaxation) is found to have the same N dependence as the inverse of the minimum positive Lyapunov exponent. The relaxation time to the final thermal equilibrium differs from the inverse of the Lyapunov exponents and the Kolmogorov-Sinai time.

PACS number(s): 05.45.Jn, 05.20.−y, 05.70.Ln, 98.10.+z

I. INTRODUCTION

Relaxation is the most fundamental process in the evolution of many-body systems. The classical statistical theory is based on ergodic property, which is considered to be established after relaxation. However, all systems do not always show such an idealistic relaxation. A historical example is the FPU (Fermi-Pasta-Ulam) problem [1], which exhibits phase space trapping and does not relax to the equipartition for a very long time [2].

One-dimensional self-gravitating sheet systems (OGS) have been known to show strange evolution for nearly thirty years. Hohl [3–5] first asserted that OGS relaxes to the thermodynamical equilibrium (the isothermal distribution) in a time scale of about $N^2 t_c$, where N is the number of sheets and t_c is typical time for a sheet to cross the system. Later, more precise numerical experiments have figured out that the Hohl's result was not right, and then other possible explanations for the relaxation time have been presented in the eighties. A Belgian group [6,7] claimed that the OGS relaxed in time shorter than Nt_c, whereas a Texas group [8,9] showed that the system showed long lived correlation and never relaxed even after $2N^2 t_c$. Tsuchiya, Gouda, and Konishi [10] (hereafter TGK) suggested that this contradiction can be resolved in view of two different types of relaxations: the *microscopic* and the *macroscopic* relaxations. At the time scale of Nt_c, cumulative effect of the mean field fluctuation makes the energies of the individual particles change noticeably. Averaging over these fluctuations one realizes that energy equipartition is indeed achieved, thus there gives the equipartition of energies, thus there is a relaxation at this time scale. By this relaxation, however, the system is led not to the thermal equilibrium but only to a quasiequilibrium. The global shape of the one-body distribution remains different from that of the thermal equilibrium. This relaxation appears only in the microscopic dynamics, thus it is called the microscopic relaxation. The global shape of the one-body distribution transforms in much longer time scale. For example, a quasiequilibrium (the water-bag distribution, which has the longest life time) begins to transform at $4\times10^4 Nt_c$ in average. TGK called this transformation the macroscopic relaxation, but later Tsuchiya, Gouda, and Konishi [11] have shown that this transformation corresponds to the onset of the *itinerant stage*. In this stage, the one-body distribution stays in a quasiequilibrium for some time and then changes to other quasiequilibrium. This transformation continues forever. Probability density of the life time of the quasiequilibria has a power law distribution with a long time cutoff and the longest life time is $\sim10^4 Nt_c$. Only by averaging over a time longer than the longest life time of the quasiequilibria, the one-body distribution becomes that of the thermal equilibrium, which is defined as the maximum entropy state. Yawn and Miller [12,13] also showed that the ergodicity is established not in $10^4 Nt_c$, but in several $10^5 Nt_c$. Therefore the time $\sim10^6 Nt_c$ is necessary for relaxation to the thermal equilibrium, and called the *thermal relaxation time*. Although there are some attempts to clarify the mechanisms of these relaxations [10,14,15,11], the reason why the system does not relax for such a long time is still unclear.

At the view of chaotic theory in dynamical systems, relaxation is understood as mixing in phase space, and its time scale is given by the Kolmogorov-Sinai (KS) time, $\tau_{\mathrm{KS}} = 1/h_{\mathrm{KS}}$, where h_{KS} is the Kolmogorov-Sinai entropy. However, it does not simply correspond to the relaxation of the one-body distribution function, which is of interest in many-body systems. Recently, Dellago and Posch [16] showed that in a hard sphere gas, the KS time equals the mixing time of neighboring orbits in the phase space, whereas the relaxation of the one-body distribution function corresponds to the collision time between particles. Now, it is fruitful to study relation between relaxation and some dynamical quantities, such as the KS entropy and the Lyapunov exponents, in the OGS. Milanović *et al.* [15] showed the Lyapunov spectrum and the Kolmogorov-Sinai entropy in the OGS for $10\leqslant N \leqslant 24$. However, since it is known that the chaotic behavior

*Present address: Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan.

changes for $N\sim30$ for the OGS [17], it is considerably important to extend the analysis to the system larger than $N\sim30$. In this paper, we extend the number of sheets to $N=256$ and follow the evolution numerically up to $T\sim10^6 N t_c$, which is long enough for the thermal relaxation [11].

In Sec. II, we introduce our model and report some parameters of the numerical simulations. The results are given in Sec. III and Sec. IV is devoted to conclusions and discussion.

II. NUMERICAL SIMULATIONS

The OGS comprises N identical plane-parallel mass sheets, each of which has uniform mass density and infinite in, say, the y and z directions. They move only in the x direction under their mutual gravity. When two of the sheets intersect, they pass through each other. The Hamiltonian of the system has the form

$$H=\frac{m}{2}\sum_{i=1}^{N} v_i^2+(2\pi G m^2)\sum_{i<j}|x_j-x_i|, \tag{1}$$

where m, v_i, and x_i are the mass (surface density), velocity, and position of the ith sheet, respectively. Since the gravitational field is uniform, the individual particles moves parabolically, until they intersect with the neighbors. Thus the evolution of the system can be followed by solving quadratic equations. This property helps us to calculate long time evolution with a high accuracy. Since length and velocity (thus also energy) can be scaled in the system, the number of the sheets N is the only free parameter. The crossing time is defined by

$$t_c=(1/4\pi G M)(4E/M)^{1/2}, \tag{2}$$

where $M=Nm$ and E is the total mass and total energy of the system. In comparing systems with different N, the mass of a sheet is proportional to $1/N$ in this scaling. Detailed descriptions of the evolution of the OGS can be found in our previous papers [18,10,11].

In order to investigate dynamical aspects of the system, we calculated the Lyapunov spectrum. The basic numerical algorithm follows Shimada and Nagashima [19], and detailed description of the procedure for the OGS can be found in Refs. [18,15]. We made numerical integration for $8\leqslant N\leqslant128$ up to $10^8 t_c$, which is enough time for the system to relax. In this integration time, convergence of the Lyapunov exponents is better than 1%, and the results do not depend on initial conditions. For $N=256$ we stopped the simulation at $1.8\times10^7 t_c$, which is about ten times less than the relaxation time. This is just a temporal data, but fluctuations of the Lyapunov exponents at the end of simulations are less than 1%, and as we will see in the next section, all quantities are just on the scaling law of the smaller N.

III. RESULTS

Figure 1 shows the spectrum of the Lyapunov exponents, $\{\lambda_i\}$, where their unit is $1/t_c$. This figure is the same diagram as Fig. 6 in Milanović *et al.* [15], but the range of N is extended to $8\leqslant N\leqslant256$. In the horizontal axis, l is the index

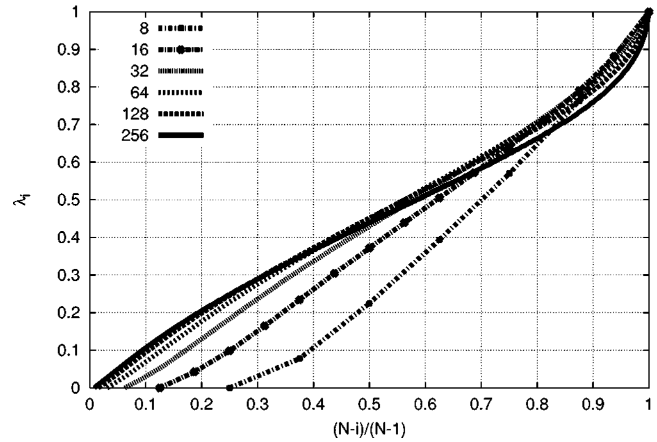

FIG. 1. Spectrum of the positive Lyapunov exponents for various N. The index of the Lyapunov exponents is scaled to 0 to 1.0. The vertical axis shows the Lyapunov exponents normalized by the value of the maximum Lyapunov exponent. The exponents are integrated up to $10^8 t_c$ for $N\leqslant128$, and $1.8\times10^7 t_c$ for $N=256$. In all cases, residual fluctuations of the exponents are less than 1%.

of the Lyapunov exponents, which is labeled in the order from the maximum to the minimum. Thus all the positive Lyapunov exponents ($i\leqslant N$) is scaled between 0 to 1 in the axis. The vertical axis shows the Lyapunov exponents normalized by the maximum Lyapunov exponents, λ_1. Milanović *et al.* [15] stated that the shape of the spectrum approximately converges for large N. A closer look, however, shows bending of the spectrum, which is most clearly seen at $(N-i)/(N-1)\sim0.9$. This bending seems to increase with N for $N\geqslant32$. Though $N=256$ is not large enough, the figure suggests that the Lyapunov spectrum does not converge for larger N.

Figure 2 shows N dependence of the maximum (λ_1), the minimum positive Lyapunov exponent (λ_{N-2}), and the KS entropy h_{KS} per the number of freedom. λ_1 is proportional to $N^{-1/5}$ for $N\geqslant32$. The similar dependence was already given in Fig. 13 in Tsuchiya *et al.* [18]. This figure, in fact, shows difference of the maximum Lyapunov exponents in different

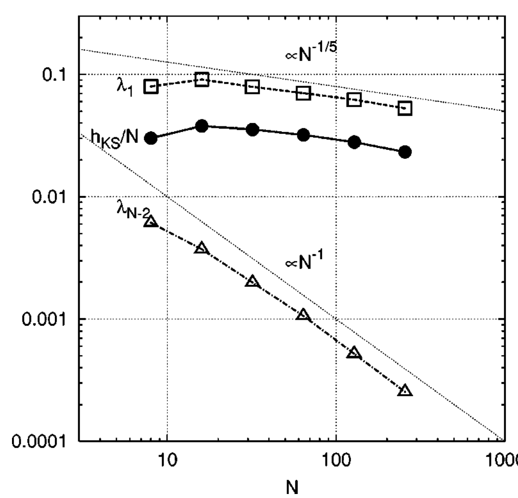

FIG. 2. Dependence of the KS entropy (solid line with the symbol ●), the maximum Lyapunov exponent (long dashed curve with the symbol □), and the minimum positive Lyapunov exponent (dashed-dotted curve with the symbol △).

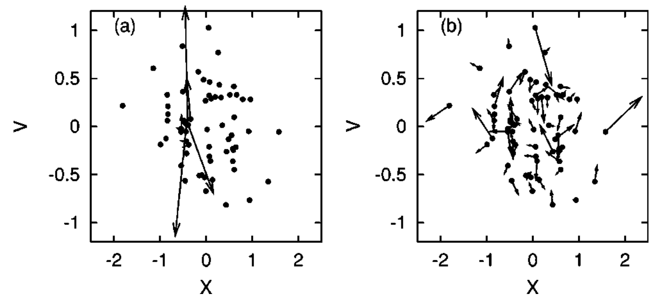

FIG. 3. Lyapunov vectors for $N=64$. Filled circles indicate positions of N sheets and the arrows give direction of the Lyapunov vector: (a) the Lyapunov vector for λ_1, (b) that for λ_{N-2}.

quasiequilibria. The quasiequilibrium appears only in a shorter time scale and those Lyapunov exponents converge to temporal values. While λ_1 of the water-bag distributions decreases as $N^{-1/4}$, that of the isothermal distributions, which represents the thermal equilibrium, decreases as $N^{-1/5}$. Decreasing nature of the Lyapunov exponent may indicate that the OGS approaches closer to an integrable system for larger N. It is very interesting that the power of $-1/5$ is different from that observed in some other systems [20,21], which is $-1/3$. In those systems the power can be explained by means of a random matrix [22] approximation, where it is unclear if such approach would be valid also for the OGS.

As expected from the spectrum the KS entropy divided by N is also proportional to $N^{-1/5}$, thus $h_{\mathrm{KS}} \propto N^{4/5}$. It is different from cases of extensive systems, where the KS entropy commonly increases linearly with N [23]. This might be because our system is fully coupled and not extensive. Our result is also different from the conjecture by Benettin *et al.* [24] that h_{KS} increases linearly with N. It is clear that the inverses of both the maximum Lyapunov exponents and the KS entropy do not give the time scale of any type of relaxation time. The same dependence for the KS entropy is reported in Latora, Rapisarda, and Ruffo [25] for a system of fully coupled Hamiltonian rotators with attractive interaction, though they stated that a more refined numerical analysis is needed to confirm these results.

The N dependence of small positive Lyapunov exponents are quite different from larger ones. In Fig. 2, the minimum positive Lyapunov exponent, λ_{N-2}, is shown by a dashed dotted line with the symbol $\triangle$. It decreases linearly for $N \geqslant 32$, and its time scale $1/\lambda_{N-2}$ is about the same as the microscopic relaxation time ($\sim N t_c$). It is worth noting that if the Lyapunov spectrum converges then the λ_{N-2} would decrease as $N^{-6/5}$, whereas the result does not. This difference is an evidence that the shape of the Lyapunov spectrum has not yet converged.

The Lyapunov vectors also give useful information about instabilities associated with the Lyapunov exponents. The Lyapunov vector for λ_i is a unit vector in the phase space, and the instability grows with the ith fastest rate in that direction. Figure 3 shows projection of the Lyapunov vector for $N=64$ on to the one-body phase space. Filled circles indicate positions of N sheets at a moment and the arrows give the direction of the Lyapunov vector at that time. The length of the vectors are scaled so as to see the direction clearly. Figure 3(a) is for the maximum Lyapunov exponent λ_1 and Fig. 3(b) is for the minimum positive one λ_{N-2}. The direction of the Lyapunov vectors change in time, but the characteristics of the instabilities are the same. The instability corresponding to λ_1 is associated to a few particles in a central small region in the phase space. Most of the particles are not affected by the instability. On the other hand, the instability corresponding to λ_{N-2} has a global character; it makes all particles mix in the phase space. This is the very effect of relaxation. These features are commonly seen for different N. Similar localization of Lyapunov vectors were reported also in coupled map lattices [26]. Hence this phenomenon might be universal.

The same N dependence of the $1/\lambda_{N-2}$ as the microscopic relaxation time, and the direction of the Lyapunov vector, may be suggesting that the microscopic relaxation time is determined by the growing time of the weakest instability, which is determined by the minimum positive Lyapunov exponent; in other words, this time is necessary for the phase space orbit to mix in the phase space in all the directions of freedom. In our working model of the evolution of the OGS [10,11], the phase space is divided by some barriers which keep the phase orbit inside for a long time. The microscopic relaxation is considered to be a diffusion process in the barriered region [10,14], and in the time $\sim N t_c$, restricted ergodicity is established within the barriered region. This time may correspond to the diffusion time in the slowest direction.

IV. CONCLUSIONS AND DISCUSSION

In the ergodic theory, the KS time represents the time scale of ''mixing'' in the phase space. On the other hand, the relaxation of the one-body distribution is of the most interest in systems with large degrees of freedom. We have shown that the time scale of the relaxation of one-body distribution (both the microscopic and thermal relaxation) is certainly different from that of the KS time, and found that the growing time of the weakest Lyapunov instability is about the same as the microscopic relaxation time. In addition, taking into account the direction of the eigenvector of the weakest Lyapunov exponent, it is suggested that the microscopic relaxation is determined by the weakest Lyapunov instability.

The KS entropy is defined as inverse of a typical time for the system to increase ''information.'' This definition does not depend on the number of degrees of freedom. In higher dimensions, however, even very small growth of instability can increase information quite rapidly. Therefore the KS time does not seem suitable to characterize the relaxation of the one-body distribution function.

The relaxation of the one-body distribution function implies that the system becomes ergodic, and the mechanism to attain ergodicity is diffusion and mixing of the phase space orbits. The diffusion time is expected to be represented by time scale of orbital instability, i.e., the Lyapunov time. For systems with many degrees of freedom, there should be difference in diffusion time for different direction in the phase space, so that time of accomplishment of ergodicity might be determined by the time of the slowest diffusion. Therefore we surmise that inverse of the smallest Lyapunov exponent represents the relaxation time.

In OGS, we presumed that the phase space is separated into small regions, which correspond to quasiequilibria, and each region has approximate ergodicity [10]. The diffusion time in the regions is the microscopic relaxation time. By integrating long time evolution, the averaged diffusion time in the regions is supposed to correspond to λ_{N-2}.

A remaining problem is why the KS time and any of the Lyapunov times do not give the much long time scale of the thermal relaxation in the OGS. In our working model, the thermal relaxation is the successive transitions of the phase space orbit among the barriered regions, which corresponds to the quasiequilibria. Actual time of the thermal relaxation is the maximum time of transition among quasiequilibria. The fact that the Lyapunov exponents do not give the correct time of thermalization indicates that the transition is due to a different mechanism from local instabilities. There are some pieces of evidence that collective effects are responsible for keeping the system in a quasiequilibrium [11,27]. This may suggest that we need a new dynamical quantity which characterizes the slow diffusion.

Existence of the long-lived quasiequilibria is reported in various systems, such as one-dimensional systems with attractive pair potential $|x_i - x_j|^\nu$, where ν is a positive parameter [15], globally coupled spin models [28,20], and a two-dimensional system with long-range forces [29]. Therefore the slow relaxation seems universal property in systems with long-range forces. It is important to clarify the mechanism of the slow relaxation to construct a new statistical mechanics of the many-body systems with long-range forces.

ACKNOWLEDGMENTS

The authors are grateful to T. Konishi for many critical suggestions. We also thank B. N. Miller and Y. Aizawa for valuable discussions. This work was supported in part by the JSPS and in part by a Grant-in-Aid for Scientific Research (Grant No.10640229) from the Ministry of Education, Science, Sports and Culture of Japan.

[1] E. Fermi, J.R. Pasta, and S. Ulam, *Collected Works of Enrico Fermi* (University of Chicago Press, Chicago, 1965), Vol. 2, p. 978.
[2] L. Casetti, M. Cerruti-Sola, M. Pettini, and E.G.D. Cohen, Phys. Rev. E **55**, 6566 (1997).
[3] F. Hohl and D.T. Broaddus, Phys. Lett. **25A**, 713 (1967).
[4] F. Hohl and M.R. Feix, Astrophys. J. **147**, 1164 (1967).
[5] F. Hohl and J.W. Campbell, Astron. J. **73**, 611 (1968).
[6] M. Luwel, G. Severne, and P.J. Rousseeuw, Astrophys. Space Sci. **100**, 261 (1984).
[7] G. Severne, M. Luwel, and P.J. Rousseeuw, Astron. Astrophys. **138**, 365 (1984).
[8] H.L. Wright, B.N. Miller, and W.E. Stein, Astrophys. Space Sci. **84**, 421 (1982).
[9] B.N. Miller and C.J. Reidl, Jr., Astrophys. J. **348**, 203 (1990).
[10] T. Tsuchiya, N. Gouda, and T. Konishi, Phys. Rev. E **53**, 2210 (1996).
[11] T. Tsuchiya, N. Gouda, and T. Konishi, Astrophys. Space Sci. **257**, 319 (1998).
[12] K.R. Yawn and B.N. Miller, Phys. Rev. E **56**, 2429 (1997).
[13] K.R. Yawn and B.N. Miller, Phys. Rev. Lett. **79**, 3561 (1997).
[14] B.N. Miller, Phys. Rev. E **53**, R4279 (1996).
[15] L. Milanović, H.A. Posch, and W. Thirring, Phys. Rev. E **57**, 2763 (1998).
[16] C. Dellago and H.A. Posch, Phys. Rev. E **55**, R9 (1997).
[17] C.J. Reidl, Jr. and B.N. Miller, Phys. Rev. E **51**, 884 (1995).
[18] T. Tsuchiya, T. Konishi, and N. Gouda, Phys. Rev. E **50**, 2607 (1994).
[19] I. Shimada and T. Nagashima, Prog. Theor. Phys. **61**, 1605 (1979).
[20] V. Latora, A. Rapisarda, and S. Ruffo, Phys. Rev. Lett. **80**, 692 (1998).
[21] A. Torcini and M. Antoni, Phys. Rev. E **59**, 2746 (1999).
[22] G. Parisi and A. Vulpiani, J. Phys. A **19**, L425 (1986).
[23] P. Grassberger, Phys. Scr. **40**, 346 (1989).
[24] G. Benettin, C. Froeschle, and J.P. Scheidecker, Phys. Rev. A **19**, 2454 (1979).
[25] V. Latota, A. Rapisarda, and S. Ruffo, Physica D **131**, 38 (1999).
[26] K. Kaneko, Physica D **23**, 436 (1986).
[27] J.L. Rouet and M.R. Feix, Phys. Rev. E **59**, 73 (1999).
[28] Y.Y. Yamaguchi, Prog. Theor. Phys. **95**, 717 (1996).
[29] M. Antoni and A. Torcini, Phys. Rev. E **57**, R6233 (1998).